# NEW ALGEBRAIC SYSTEMS FLOWER, GARDEN and FARM

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## ABSTRACT

**In this paper we introduced the notions of: ATL law, RAL law, Semiflower, Flower, Garden and Farm.**

**Some new algebraic concepts have been defined. An algorithm for ATL test has been explained.**

**Some Lemmas, Propositions and Theorems have been proved.**

**Key Words**: Semigroup, Group, Ring Field, Semiflower Flower, Garden, Farm.

# بنى جبرية حديثة
# الزهرة ، الحديقة ، المزرعة

**أنور توفيق اللحام**

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## الملخص

**أدخلت في هذا البحث المفاهيم الجبرية الحديثة التالية:**

**خاصية ATL وخاصية RAL ونصف الزهرة والزهرة والحديقة والمزرعة.**

**كما شرحت خوارزمية لفحص الخاصية ATL لعملية إثنانية (قانون تشكيل) معرفة على مجموعـة غير خالية بجدول.**

**كما تم إثبات عدد من التمهيديات والقضايا والمبرهنات المتعلقة بهذه البنى الجبرية الحديثة.**

**الكلمات المفتاحية:** نصف الزمرة، الحلقة، نصف الزهرة، الزهرة، الحديقة، المزرعة.

# 1. THE FLOWER

## 1-1 BASIC DEFINITIONS

DEFINITION ( 1 ) A groupoid is an ordered pair $(S, \star)$ where S is a non empty set and $\star$ is a binary operation on S. [1]

DEFINITION ( 2 ) A binary operation $\star$ on a non empty set S is ATL if for all $a,b,c \in S$

$$a \star (b \star c) = c \star (b \star a)$$

DEFINITION ( 3 ) A binary operation $\star$ on a non empty set S is RAL if for all $a,b,c \in S$

$$(a \star b) \star c = (a \star c) \star b$$

DEFINITION ( 4 ) A semiflower is an ordered pair $(S, \star)$ where S is a non empty set and $\star$ is an ATL binary operation on S.

DEFINITION ( 5 ) A flower S is a semiflower with a right identity e, such that

$$a \star a = e \qquad \forall a \in S.$$

Remark We can give the following equivalent definition for the flower:

DEFINITION ( 6 ) A flower is an ordered pair $(S, \star)$ where S is a non empty set and $\star$ is a binary operation on S satisfying the following axioms:

(i) $a \star (b \star c) = c \star (b \star a) \quad \forall a, b, c \in S$ (ATL law)

(ii) there exists an element e in S such that

$a \star e = a \quad \forall a \in S$ (e is a right identity of S)

(iii) $a \star a = e \quad \forall a \in S$

DEFINITION ( 7 ) A nonempty subset H of a flower $(S, \star)$ which is a flower under the operation $\star$ restricted to H is called a subflower of S.

DEFINITION ( 8 ) A group $(G, \star)$ is called Lahhamian group if :

$$a \star a = e \qquad \forall a \in G \qquad \text{where e is the identity of G.}$$

N.B In the following we shall become less formal and say S is a flower and write ( ab ) instead of ( $a \star b$ ).

1-2 EXAMPLES

( 1 ) Z, Q, R, and C are flowers under the usual subtraction ( – ) with $e = 0$.

( 2 ) $Q^* = Q - \{0\}$, $R^*$, $C^*$, $Q^+ = \{q \in Q : q > 0\}$, and $R^+$ are flowers under the usual division ($\div$) with $e = 1$.

( 3 ) The following two tables define two finite flowers:

| ⋆ | e | a | b | c | d |
|---|---|---|---|---|---|
| e | e | d | c | b | a |
| a | a | e | d | c | b |
| b | b | a | e | d | c |
| c | c | b | a | e | d |
| d | d | c | b | a | e |

| ⋆ | e | a | b | c | d | f |
|---|---|---|---|---|---|---|
| e | e | c | d | a | b | f |
| a | a | e | b | c | f | d |
| b | b | d | e | f | a | c |
| c | c | a | f | e | d | b |
| d | d | f | c | b | e | a |
| f | f | b | a | d | c | e |

1-3 ATL TEST

To test a finite groupoid ( S , ⋆ ) for ATL law, when the binary operation ⋆ is defined by a table,is usually quite a tedious business. I suggest the following algorithm for test.

The procedure is to be carried out for each element $a$ of the groupoid S.
Consider the operation ∘ defined in S as follows: $x \circ y = x \star (a \star y) \quad \forall x, y \in S$
ATL law holds in $(S, \star)$ if and only if, for each fixed element $a \in S$, this binary operation ∘ is commutative.

The idea is essentially to construct the table for ∘ , for each element $a \in S$, and then see if ∘ is commutative or not.

The ∘ – table is obtained from the original ⋆ – table by replacing, for each $y \in S$, the $y$ column by the $a \star y$ column.

For convenience in performing the test, we replace the top index line of the ∘ *–table* by the $a$ row of the ⋆ *–table*. For each entry $a \star y$ in the $a$ row of the ⋆ *–table* tells us what column of the ⋆ *–table* to copy down as the $y$ column of the ⋆ *–table* .

## 1-4 MAIN RESULTS

LEMMA 1. A nonempty subset H of a flower ( S, $\star$ ) is a subflower if and only if H is closed under the binary operation $\star$.

PROOF obvious.

LEMMA 2. If ( S, $\star$ ) is a flower then the following are true $\forall x,y,z \in S$ :

( 1 ) $x(xy) = y$
( 2 ) $yx = e(xy)$
( 3 ) $e(xy) = (ex)(ey)$
( 4 ) $xy = x$ if and only if $y = e$
( 5 ) $xy = e$ if and only if $x = y$
( 6 ) $(xy)x = ey$
( 7 ) $(xe)x = x(xe) = e$

PROOF

(1) $x(xy) = y(xx)$ ATL law
$= ye = y$ axioms 2 and 3
(2) $e(xy) = y(xe) = yx$ ATL law and axiom 2
(3) $(ex)(ey) = y(e(ex)) = yx = e(xy)$ ATL, ( 1 ), and ( 2 )
(4) (i) $xy = x \Rightarrow x(xy) = xx \Rightarrow y = e$

(ii) $y = e \Rightarrow xy = x$

(5) (i) $xy = e \Rightarrow x(xy) = xe \Rightarrow y = x$
(ii) $x = y \Rightarrow xy = yy \Rightarrow xy = e$
(6) Let $(xy)x = u$ then $e((xy)x) = eu \Rightarrow x(xy) = eu \Rightarrow y(xx) = eu$
$\Rightarrow y = eu \Rightarrow e\,y = e(eu) = ue = u$
(7) (i) $(xe)x = xx = e$
(ii) $x(xe) = xx = e$

COROLLARY 1. Let $(S,\star)$ be a flower, then :

(i) S has a unique right identity.
(ii) every element of S has a unique right inverse in S.

LEMMA 3. If $(S,\star)$ is a flower then the following are true $\forall x,y,z \in S$ :

(1) $(xy)(xz) = zy$
(2) $((xy)(xz))(zy) = e$ ( e is the right identity of S )
(3) $(x(yx))(xy) = x$
(4) $(xy)z = x(y(ez))$

PROOF

(1) $(xy)(xz) = z(x(xy)) = zy$ — ATL law and Lemma 2

(2) $((xy)(xz))(zy) = (z(x(xy))(zy) = (zy)(zy) = e$ — ATL law and axiom 3

(3) $(x(yx))(xy) = y(x(x(yx)) = y((yx)(xx))$

$= y((yx)e) = y(yx) = x$ — ATL law and Lemma 2

(4) $(xy)z = e(z(xy)) = (ez)(e(xy))$

$= (ez)(yx) = x(y(ez))$ — ATL law and Lemma 2

PROPOSITION 1. If $(S,\star)$ is a flower then :

$$(b\star c)\star a = (b\star a)\star c \qquad \forall a,b,c\in S$$

PROOF

S is a flower implies that $a\star(b\star c) = c\star(b\star a) \qquad \forall a,b,c\in S$

Let $e$ be the right identity of S then $e\star(a\star(b\star c)) = e\star(c\star(b\star a))$

So $(b\star c)\star a = (b\star a)\star c \qquad \forall a,b,c\in S$

PROPOSITION 2. If ( G , . ) is an abilian group, then G with the new binary operation $\star$ defined as follows: $a\star b = a.b^{-1} \quad \forall a,b\in G$ is a flower.

PROOF

(i) $a\star(b\star c) = a\star(bc^{-1}) = a(bc^{-1})^{-1} = a(cb^{-1}) = c(b^{-1}a) = c(ba^{-1})^{-1}$

$= c\star(ba^{-1}) = c\star(b\star a) \qquad \forall a,b,c\in G$

(ii) $a\star e = a \quad \forall a\in G$ ( $e$ is the identity of the group G )

(iii) $a\star a = aa^{-1} = e \qquad \forall a\in G$

N.B. ( G , $\star$ ) is not a group in general.

EXAMPLES ( 4 ) $(Z,+)$ is an abelian group, so $(Z,\star)$ is a flower, but it is not a group.

( 5 ) The Klein 4-group (Viergruppe), $V_4$, is an abelian group and it is a group also with $\star$.

( 6 ) $(Z_p^*, .)$ is an abelian group, ( $p$ is a prime number), so $(Z_p^*,\star)$ is a flower but it is not a group.

PROPOSITION 3. If G is a Lahhamian group then :

( i ) G is commutative.

( ii ) G is a flower.

PROOF

( i ) $a,b \in G \Rightarrow (ab)(ba) = a(bb)a = e$ ( $e$ is the identity of $G$ )

Which implies that $ab = ba$ $(\forall a,b \in G)$ because the inverse is unique.

( ii ) Obvious.

THEOREM 1. Let S be a flower, then the left and right cancellation laws hold in S.

PROOF

Let e be the right identity of S and let $x,y,z \in S$ then:

(i) if $xy = xz$ then $x(xy) = x(xz)$ which implies $y = z$

(ii) if $yx = zx$ then $(yx)(ex) = (zx)(ex)$

$x(e(yx)) = x(e(zx))$ (ATL law)

$x(xy) = x(xz) \Rightarrow y = z$ (Lemma 2)

THEOREM 2. Let S be a flower and let $a,b \in S$. The equations $ax = b$ and $ya = b$ have unique solutions for $x,y \in S$.

PROOF

Let e be the right identity of S

(i) $ax = b \Rightarrow a(ax) = ab \Rightarrow x = ab$ (Lemma 2)

(ii) $ya = b \Rightarrow (ya)(ea) = b(ea) \Rightarrow a(e(ya)) = b(ea)$ (Lemma 2)

$a(ay) = b(ea) \Rightarrow y = b(ea)$ (Lemma 2)

Now the uniqueness can be proved easily because S is cancellative.

THEOREM 3. Let S be a flower then:

(1) $aS = Sa = S$ $\forall a \in S.$

(2) $S^2 = S$

PROOF

(1) Let $a$ be any element of S , $x \in S \Rightarrow a(ax) = x \Rightarrow x \in aS \Rightarrow S \subseteq aS.$

2 But $aS \subseteq S$ so $aS = S.$ The same for $Sa = S.$

(2) Let $x \in S$ then $\forall t \in S$ $x = t(tx) \in S^2 \Rightarrow S \subseteq S^2$

But $S^2 \subseteq S$ therefore $S^2 = S$

DEFINITION 9. A function $\rho$ $[\lambda]$ from a flower S to S is a right (left) translation of S if

$$\rho(xy) = x(\rho(y)) \quad [\lambda(xy) = \lambda(x)y] \qquad \forall x,y \in S$$

THEOREM 4. Let S be a flower, then every right translation of S commutes with every left translation of S.

PROOF

We have $S^2 = S$ so for every $z \in S$ there exists $x,y \in S$ such that $z = xy$

Now let $\rho : S \to S$ be a right translation of S,
$\lambda : S \to S$ be a left translation of S,

then $\forall z \in S$

$$(\lambda \circ \rho)(z) = \lambda(\rho(xy)) = \lambda(x\rho(y)) = \lambda(x)\rho(y)$$

$$(\rho \circ \lambda)(z) = \rho(\lambda(xy)) = \rho(\lambda(x)(y)) = \lambda(x)\rho(y)$$

Therefore $\rho \circ \lambda = \lambda \circ \rho$

THEOREM 5. A flower $( S , \star )$ is a Lahhamian group if and only if it is associative.

PROOF
(i) If S is a group then it is associative.
(ii) If S is associative then it satisfies the three axioms of the group, and so it is a Lahhamian group.

THEOREM 6. A flower $( S , \star )$ is a Lahhamian group if and only if it is commutative.

PROOF

(i) If S is commutative then :
$a \star (b \star c) = c \star (b \star a) \quad \forall a,b,c \in S \;\Rightarrow\; a \star (b \star c) = (a \star b) \star c \quad \forall a,b,c \in S$
then S is associative, and so it is a Lahhamian group.
(ii) If S is a group then it has an identity e and $\forall x,y \in S$
$x \star y = e \star (x \star y) = y \star x$ Lemma 2

THEOREM 7. A flower $( S , \star )$ is a Lahhamian group if and only if it has an identity.

PROOF
(i) If S is a group then it has an identity.
(ii) If S has an identity e then $\forall x,y \in S$ $\quad x \star y = e \star (x \star y) = y \star x$ by Lemma 2
so S is commutative, and therefore it is a group by theorem 6.

As a corollary of the last three theorems we can write the following theorem:

THEOREM 8. Let $( S , \star )$ be a flower, then the following conditions are equivalent:
( 1 ) S is a Lahhmian group.
( 2 ) S has an identity.
( 3 ) S is commutative.

( 4 ) S is associative.

EXAMPLE 7. Let A be a finite set, ($A \neq \phi$), and $S = P(A)$ be the set of all parts of A, then ( S , $\triangle$ ),( where $\triangle$ is the symmetric difference ) is a Lahhamian group of order $2^n$ ; ( $n = |A|$ ).

As a corollary of example 7 we can write:

PROPOSITION 4. For any $n \in N$ there exists a Lahhamian group of order $2^n$.

PROPOSITION 5. There is no Lahhamian group of prime order greater than 2.

PROOF

Let G be a Lahhamian group of prime order $p \geq 3$ then G is isomorphic to $(Z_p, +)$, which is obviously not a Lahhamian group.

THEOREM 9. (Cauchy) If G is a finite group and p is a prime dividing $|G|$, then G has an element of order p. [2]

THEOREM 10 Every finite Lahhamian group G has order $2^n$ such that $n \in N$.

PROOF

Let $p$ be a prime ($p \geq 3$), such that $p$ divides the order of G, then by Cauchy theorem, G has an element $x \in G$ of order $p$, which implies that $x^2 \neq 1$, ( where 1 is the identity of G ), which is a contradiction.

## 2 - THE GARDEN

DEFINITION 10. A garden is a triple $(S, \star, \diamond)$, where S is a non empty set has at least two elements, $\star$ and $\diamond$ are two binary operations on S, ( $\diamond : S \times S^* \to S$ ) and $S^* = S - \{0\}$, ( 0 is the right identity in $(S, \star)$ ), satisfies the following axioms:

( 1 ) $(S, \star)$ is a flower.

( 2 ) $\diamond$ is ATL ( i.e. $a \diamond (b \diamond c) = c \diamond (b \diamond a) \quad \forall a, b, c \in S^*$ )

( 3 ) The distributive law

$(a \star b) \diamond c = (a \diamond c) \star (b \diamond c) \quad \forall a, b \in S \quad \forall c \in S^*$ holds in S.

EXAMPLE 8. Let A be a non empty set, P( A ) is the set of all its parts, then $(P(A), \triangle, \cap)$ is a garden.

EXAMPLE 9. Q, R, and C are gardens under the usual two binary operations $-$ and $\div$.

LEMMA 4. Let $(S, \star, \diamond)$ be a garden, then $\forall x \in S$ and $\forall y \in S^*$

(i) $0 \diamond y = 0$ .

(ii) $0 \star (x \diamond y) = (0 \star x) \diamond y$

(iii) $(0 \star x) \diamond (0 \star y) = 0 \star (x \diamond (0 \star y))$

PROOF

(i) We have $(a \star b) \diamond y = (a \diamond y) \star (b \diamond y) \quad \forall a,b \in S \quad \forall y \in S^*$

Let $a = b$ then $0 \diamond y = (a \diamond y) \star (a \diamond y) = 0 \quad \forall y \in S^*$

(ii) $(0 \star x) \diamond y = (0 \diamond y) \star (x \diamond y) = 0 \star (x \diamond y) \quad \forall x \in S \quad \forall y \in S^*$

(iii) $(0 \star x) \diamond (0 \star y) = (0 \diamond (0 \star y)) \star (x \diamond (0 \star y))$
$= 0 \star (x \diamond (0 \star y)) \quad \forall x \in S \quad \forall y \in S^*$

THEOREM 11. A garden $(S, \star, \diamond)$ is a ring if and only if $\star$ and $\diamond$ are commutatives.

PROOF

(i) If $\star$ is commutative then $(S, \star)$ is a group ( by theorem 6 ).
If $\diamond$ is commutative then it is associative.
Hence S is a ring.

(ii) If S is a ring then $\star$ and $\diamond$ are commutatives ( theorem 8).

THEOREM 12. A garden $(S, \star, \diamond)$ of order n divisible by an odd prime is not a ring.

PROOF

Let n be divisible by a prime $p \geq 3$ then $(S, \star)$ can't be a group ( by theorem 10 ).

3 - THE FARM

DEFINITION 11. A farm is a triple $(S, \star, \diamond)$, where S is a set has at least two elements, $\star$ and $\diamond$ are two binary operations on S, ( $\diamond : S \times S^* \to S$ ) and

$S^* = S - \{0\}$, ( 0 is the right identity in $(S, \star)$ ),
satisfies the following axioms:

( 1 ) $(S, \star)$ is a flower.
( 2 ) $(S^*, \diamond)$ is a flower.
( 3 ) The distributive law
$(a \star b) \diamond c = (a \diamond c) \star (b \diamond c) \quad \forall a,b \in S \quad \forall c \in S^*$ holds in S.

Or shortly we can say:

DEFINITION 12 A farm $(S, \star, \diamond)$ is a garden such that $(S^*, \diamond)$ is a flower.

THEOREM 13. A finite farm $(S, \star, \diamond)$, such that $|S| > 2$, is not a field.

PROOF

Let $(S, \star)$ be a group, then it is a L group, which implies that it has order $2^n$, $(n \in N)$, then $|S^*| = 2^n - 1$, then $(S^*, \diamond)$ can't be a group ( by theorem 10).

REMARK

I think that the following proposition is right, but I have not the proof yet, so I left it as a

CONJECTURE If $(S, \star)$ is a groupoid with a right identity element e,
such that $x \star x = e \quad \forall x \in S$ then for all elements $a, b, c \in S$:
$a \star (b \star c) = c \star (b \star a)$ if and only if $(a \star b) \star c = (a \star c) \star b$

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